Fractional generalizations of

LEIBNIZ' FORMULA

Nicholas Wheeler, Reed College Physics Department
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Introduction. From its name we know already that "Leibniz' Formula"

$$D^n(f\cdot g)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)} \tag{1}$$

is as ancient as it is pretty. In the case $n=1$ we recover the primitive "product rule"

$$\tfrac{d}{dx}(f\cdot g)=\tfrac{d}{dx}f\cdot g+f\cdot\tfrac{d}{dx}g \tag{2}$$

and it is, of course, by iteration of (2) that one obtains (1). One might organize the argument as follows: let (2) be notated

$$D(f\cdot g)=f(\overleftarrow{D}+\overrightarrow{D})g$$

Then

$$\begin{aligned}
D^2(f\cdot g)&=D\big\{f\overleftarrow{D}\cdot g+f\cdot\overrightarrow{D}g\big\}\\
&=f\overleftarrow{D}(\overleftarrow{D}+\overrightarrow{D})g+f(\overleftarrow{D}+\overrightarrow{D})\overrightarrow{D}g\\
&=f(\overleftarrow{D}+\overrightarrow{D})^2g\quad\text{because }\overleftarrow{D}\text{ and }\overrightarrow{D}\text{ commute}\\
&\;\;\vdots\\
D^n(f\cdot g)&=f\,\underbrace{(\overleftarrow{D}+\overrightarrow{D})^n}\,g \qquad (3)\\
&\qquad\qquad=\sum_{k=0}^{n}\binom{n}{k}\overleftarrow{D}^k\,\overrightarrow{D}^{n-k}
\end{aligned}$$

which reproduces (1).

Integration of (2) yields the "integation by parts" identity

$$f(x)g(x)\Big|_a^b=\int_a^b f'(x)g(x)\,dx+\int_a^b f(x)g'(x)\,dx$$

which in its most frequently encountered formulation might be abbreviated

$$\int fDg = \text{boundary term } - \int gDf \tag{4}$$

Recently—in connection with an early draft of the material reported in §10 of "Construction and Physical Application of the Fractional Calculus"—I for a while imagined myself to stand in need of an identity of the form

$$\int fD^n g = \text{boundary term } + \text{factor} \cdot \int gD^n f \tag{5}$$

Such an identity I did, in fact, manage to extract from Leibniz' generalization of (2)—only to discover that my imagined need was unreal. My primary purpose in this note is to record a rather pretty result what would otherwise be consigned to my wastebasket. I will take opportunity of the occasion also to inquire more deeply into the fractional analog of Leibniz' formula than was possible within the compass of the seminar notes just cited. The tail will wag the dog.

1. Integration by parts in higher integral order. In order to expose most plainly both the problem and my plan of attack, I look first to the case $n = 2$. By Leibniz' formula

$$fD^2g = D^2[fg] - 2Df \cdot Dg - gD^2 f$$

where it is the central term on the right that stands in the way of our achieving an instance of (5). But

$$Df \cdot Dg = D[gDf] - gD^2 f$$

so we have

$$\begin{aligned} fD^2g &= D^2[fg] - 2\big\{D[gDf] - gD^2f\big\} - gD^2 f \\ &= D^2[fg] - 2D[gDf] + gD^2 f \\ &= D[fDg - gDf] + gD^2 f \end{aligned}$$

giving

$$\int_a^b fD^2g = [fDg - gDf]_a^b + \int_a^b gD^2 f$$

which does indeed have the desired structure. Enlarging upon the preceeding line of argument, we in the case $n = 3$ have

$$\begin{aligned} fD^3g = D^3[fg] - &\binom{3}{1} D[Dg \cdot Df] \\ &- \left\{\binom{3}{2} - \binom{3}{1}\right\} D[D^2 f] \\ &- \left\{\binom{3}{3} - \binom{3}{2} + \binom{3}{1}\right\} gD^3 f \end{aligned}$$

giving

$$\int_a^b fD^3g = \text{boundary term } + \text{factor} \cdot \int_a^b gD^3f$$

where

$$\begin{aligned}\text{boundary term} &= \Big[D^2[fg] - \binom{3}{1}[Dg\cdot Df] - \Big\{\binom{3}{2} - \binom{3}{1}\Big\}[D^2f]\Big]_a^b \\ \text{factor} &= -\Big\{\binom{3}{3} - \binom{3}{2} + \binom{3}{1}\Big\}\end{aligned}$$

To further consolidate my sense of pattern I look finally to the case $n = 4$:

$$\begin{aligned}fD^4g = D^4[fg] - &\binom{4}{1}D[D^2g\cdot Df] \\ &-\Big\{\binom{4}{2} - \binom{4}{1}\Big\}D[Dg\cdot D^2f] \\ &-\Big\{\binom{4}{3} - \binom{4}{2} + \binom{4}{1}\Big\}D[g\cdot D^3f] \\ &-\Big\{\binom{4}{4} - \binom{4}{3} + \binom{4}{2} - \binom{4}{1}\Big\}gD^4f \\ = D^4[fg] - &\sum_{p=1}^{3}\Big\{\sum_{k=0}^{p-1}(-)^k\binom{4}{p-k}\Big\}D[D^{3-p}g\cdot D^pf] \\ - &\quad\Big\{\sum_{k=0}^{3}(-)^k\binom{4}{4-k}\Big\}gD^4f \\ = D^4[fg] - &\sum_{p=1}^{3}\Big\{\sum_{q=1}^{p}(-)^{p-q}\binom{4}{q}\Big\}D[D^{3-p}g\cdot D^pf] \\ - &\quad\Big\{\sum_{q=1}^{4}(-)^{4-q}\binom{4}{q}\Big\}gD^4f\end{aligned}$$

But[1]

$$\sum_{q=0}^{p}(-)^{p-q}\binom{n}{q} = \binom{n-1}{p} \quad : \quad p = 0,1,2,\ldots,n-1$$

so we can write

$$\begin{aligned}\sum_{q=1}^{p}(-)^{p-q}\binom{4}{q} &= \quad\sum_{q=0}^{p}(-)^{p-q}\binom{4}{q} - (-)^p = \binom{4-1}{p} - (-)^p \quad : \quad p = 1,2,3 \\ \sum_{q=1}^{4}(-)^{4-q}\binom{4}{q} &= -\sum_{q=0}^{3}(-)^{3-q}\binom{4}{q} - (-)^4 + 1 = 1 - \binom{4-1}{3} - (-)^4 \\ &= (-)^{4-1}\end{aligned}$$

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[1] See Abramowitz & Stegun, p. 822.

The clear implication is that we can in the general case write

$$fD^n g = D\Big[D^{n-1}[fg] - \sum_{p=1}^{n-1}\Big\{\binom{n-1}{p} - (-)^p\Big\}D^{n-1-p}g\cdot D^p f\Big] + (-)^n g D^n f$$

Thus do we achieve (5) with

$$\left.\begin{aligned}\text{boundary term} &= \Big[D^{n-1}[fg] - \sum_{p=1}^{n-1}\Big\{\binom{n-1}{p} - (-)^p\Big\}D^{n-1-p}g\cdot D^p f\Big]_a^b\\ \text{factor} &= (-)^n\end{aligned}\right\}\tag{6}$$

Though I originally acquired interest in (5) in connection with a problem rooted in the calculus of variations, I note in passing that it is evocative of the train of thought that leads to the invention of the concept of "self-adjointness" in operator theory. For example, in Sturm-Liouville theory[2] one says of the second-order differential operator

$$\mathsf{D} \equiv A(x)D^2 + B(x)D + C(x)$$

that it is self-adjoint if and only if there exist functions $\omega_{ij}(x)$ such that

$$\begin{aligned} f\mathsf{D}g - g\mathsf{D}f &= \tfrac{d}{dx}\Omega\\ \Omega &\equiv \begin{pmatrix} f\\ f'\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{pmatrix}\begin{pmatrix} g\\ g'\end{pmatrix}\end{aligned}$$

2. Remarks concerning the fractional extension of Leibniz' formula. The formal essentials of (3) can, if we wish, be notated

$$D^n = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\overleftarrow{D}^k\,\overrightarrow{D}^{n-k}\tag{7}$$

where

$$\begin{bmatrix} n\\ k\end{bmatrix} \equiv \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n+1-k)}\tag{8}$$

We can, in place of (7), write

$$D^n = \sum_{k=0}^{\infty}\begin{bmatrix} n\\ k\end{bmatrix}\overleftarrow{D}^k\,\overrightarrow{D}^{n-k}\tag{9}$$

[2] See QUANTUM MECHANICS (1967), Chapter II, p. 39 and references there cited.

since $\Gamma(n-k+1)$ becomes singular at $k = n+1, n+2, n+3, \ldots$, and its reciprocal therefore vanishes. Equation (7) makes formal sense *even when n is not an integer* (which I will emphasize by notational adjustment $n \mapsto \nu$), but in such cases the "Γ-switch" is deactivated; the finite series becomes an infinite series...as it does also when $n = -1, -2, -3, \ldots$, though in cases of the latter sort the argument is a bit more intricate; it hinges not (as formerly) on the absence of a singularity, but on the presence of competing singularities, since on its face (8.1) produces $\frac{\infty}{\infty}$ when n is a negative integer: one has[3] —even at the singular points of $\Gamma(x)$—

$$\frac{\Gamma(x)}{\Gamma(x-k)} = (-)^k(1-x)_k \quad : \quad k = 0, 1, 2, \ldots$$

where

$$\begin{aligned}(z)_0 &\equiv 1\\ (z)_k &\equiv z(z+1)(z+2)\cdots(x+k-1) \quad : \quad k = 1, 2, 3, \ldots\end{aligned}$$

serve to define the so-called "Pochhammer polynomials."

From (9) it follows formally that

$$D^\nu(f \cdot g) = \sum_{k=0}^{\infty} \begin{bmatrix}\nu\\ k\end{bmatrix} f \overleftarrow{D}^k\, \overrightarrow{D}^{\nu-k} g \tag{10}$$

Such a "Leibniz' formula for fractional derivatives" is, in fact, standard to the fractional calculus literature,[4] and possess a number of notable features:

- The formula retains its validity even for $\nu < 0$;
- The formula treats $f(x)$ and $g(x)$ asymmetrically;
- In the terms with $k > \nu$ the function $g(x)$ is subjected (not to a differentiation process but) to an integration process.

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[3] See J. Spanier & K. Oldham, *Atlas of Functions* (1987), Chapter 17 and **43:5:11**.

[4] See K. Oldham & J. Spanier, *The Fractional Calculus* (1974) §5.5 and K. Miller & B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (1993) Chapter IV §4. The topic has been studied closely in a series of papers by T. J. Osler: "Leibniz rule for fractional derivatives and an application to infinite series," SIAM J. Appl. Math. **18**, 658 (1970); "The fractional derivative of a composite function," SIAM J. Math. Anal. **1**, 288 (1970); "Fractional derivatives and Leibniz' rule." Amer. Math. Monthly **78**, 645 (1971); "The integral analog of Leibniz' rule," Math. Comp. **26**, 903 (1972); J. L. Lavoie, R. Tremblay & T. J. Osler, "Fractional derivatives and special functions," SIAM Rev. **18**, 240 (1976). For reasons not clear to me, the authors of the monographs cited above pay only glancing attention to the work of Osler, though it is pretty work of remarkable power—as I shall demonstrate.

To gain a sharper feeling for the implications of the preceding remarks, I look to concrete examples, from which I attempt to draw general lessons.

Useful information can, in favorable cases, be gained by the simple expedient of setting $f(x) = e^{ax}$, for then (10) reads

$$D^{\nu}\big(e^{ax}g(x)\big) = e^{ax} \cdot \sum_{k=0}^{\infty} a^k \begin{bmatrix} \nu \\ k \end{bmatrix} D^{\nu-k}g(x) \tag{11}$$

Suppose, for example, we set $\nu = \frac{1}{2}$ and take $g(x)$ to be the

$$\text{unit function} \quad : \quad u(x) = 1 \quad (\text{all } x)$$

By definition (at least within the standard Riemann-Liouville formulation of the fractional calculus)

$$D^{\frac{1}{2}}f(x) \equiv D \cdot D^{-\frac{1}{2}}f(x) \equiv D \cdot \left\{ \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{1}{\sqrt{x-y}} f(y)\,dy \right\}$$

so (according to *Mathematica*)

$$\begin{aligned} D^{\frac{1}{2}}e^{ax} &= D \cdot \left\{ \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{1}{\sqrt{x-y}} e^{ay}\,dy \right\} \quad : \quad x > 0 \text{ and } \Re(a) > 0 \\ &= D \cdot \left\{ \frac{e^{ax}\mathrm{erf}(\sqrt{ax}\,)}{\sqrt{a}} \right\} \\ &= \tfrac{1}{\sqrt{\pi x}} + \sqrt{a}e^{ax}\mathrm{erf}(\sqrt{ax}\,) \\ &= e^{ax}\Big\{ \tfrac{1}{\sqrt{\pi x}} e^{-ax} + \sqrt{a}\,\mathrm{erf}(\sqrt{ax}\,) \Big\} \\ &= e^{ax} \cdot \tfrac{1}{\sqrt{\pi x}} \Big\{ 1 + a \cdot x - a^2 \cdot \tfrac{1}{6}x^2 + a^3 \cdot \tfrac{1}{30}x^3 \\ &\qquad\qquad - a^4 \cdot \tfrac{1}{168}x^4 + a^5 \cdot \tfrac{1}{1080}x^5 - a^6 \cdot \tfrac{1}{7920}x^6 + \cdots \Big\} \end{aligned}$$

while the right side of (11) supplies

$$\begin{aligned} &= e^{ax} \cdot \sum_{k=0}^{\infty} a^k \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix} D^{\frac{1}{2}-k}u(x) \\ &= e^{ax} \cdot \Big\{ 1D^{\frac{1}{2}}u + a \cdot \tfrac{1}{2}D^{-\frac{1}{2}}u - a^2 \cdot \tfrac{1}{8}D^{-\frac{3}{2}}u + a^3 \cdot \tfrac{1}{16}D^{-\frac{5}{2}}u \\ &\qquad - a^4 \cdot \tfrac{5}{128}D^{-\frac{7}{2}}u + a^5 \cdot \tfrac{7}{256}D^{-\frac{9}{2}}u - a^6 \cdot \tfrac{21}{1024}D^{-\frac{11}{2}}u + \cdots \Big\} \end{aligned}$$

Consistency entails

$$
\begin{aligned}
D^{+\frac{1}{2}}u &= \tfrac{1}{\sqrt{\pi x}} \\
D^{-\frac{1}{2}}u &= \tfrac{1}{\sqrt{\pi x}}2x &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)}{1} \\
D^{-\frac{3}{2}}u &= \tfrac{1}{\sqrt{\pi x}}\tfrac{4}{3}x^2 &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)^2}{1\cdot 3} \\
D^{-\frac{5}{2}}u &= \tfrac{1}{\sqrt{\pi x}}\tfrac{8}{15}x^3 &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)^3}{1\cdot 3\cdot 5} \\
D^{-\frac{7}{2}}u &= \tfrac{1}{\sqrt{\pi x}}\tfrac{16}{105}x^4 &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)^4}{1\cdot 3\cdot 5\cdot 7} \\
D^{-\frac{9}{2}}u &= \tfrac{1}{\sqrt{\pi x}}\tfrac{32}{945}x^5 &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)^5}{1\cdot 3\cdot 5\cdot 7\cdot 9} \\
D^{-\frac{11}{2}}u &= \tfrac{1}{\sqrt{\pi x}}\tfrac{64}{10395}x^6 &&= \tfrac{1}{\sqrt{\pi x}}\tfrac{(2x)^6}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11} \\
&\ \vdots
\end{aligned}
$$

It is gratifying to observe that the preceding line of argument leads in the general case to a result

$$
\begin{aligned}
D^{-(k-\frac{1}{2})}x^0 &= \tfrac{1}{\sqrt{\pi x}}\tfrac{(k-1)!2^{k-1}(2x)^k}{(2k-1)!} \\
&= \tfrac{1}{\Gamma(k+\frac{1}{2})}x^{k-\frac{1}{2}}
\end{aligned}
$$

that can be read as a special case of Lacroix' construction[5]

$$
D^{-\nu}x^p = \tfrac{\Gamma(1+p)}{\Gamma(1+p+\nu)}x^{p+\nu} \tag{12.1}
$$

which—though introduced in 1819 by S. F. Lacroix on quite other (casually informal) grounds—is itself an immediate implication of the definition put forward by Liouville & Riemann:

$$
\begin{aligned}
D^{-\nu}f(x) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-y)^{\nu-1}f(y)\,dy \\
&\downarrow \\
D^{-\nu}\ x^p &= \frac{1}{\Gamma(\nu)}\int_0^x (x-y)^{\nu-1}y^p\,dy \\
&= \tfrac{\Gamma(1+p)}{\Gamma(1+p+\nu)}x^{p+\nu} \quad : \quad x>0,\ \Re(p)>-1,\ \Re(\nu)>0
\end{aligned}
$$

If we retain $g(x) = u(x) \equiv 1$ but take $f(x) = x^p$ and (for expository convenience) assume $0<\nu<1$ then Leibniz' formula (10) gives

$$
\begin{aligned}
D^\nu(x^p\cdot 1) &= \sum_{k=0}^{p}\begin{bmatrix}\nu\\k\end{bmatrix}\frac{p!}{(p-k)!}x^{p-k}\cdot D^{\nu-k}u \\
&= \begin{bmatrix}\nu\\0\end{bmatrix}x^p\cdot D^\nu u + \sum_{k=1}^{p}\begin{bmatrix}\nu\\k\end{bmatrix}\frac{p!}{(p-k)!}x^{p-k}\cdot D^{-(k-\nu)}u
\end{aligned}
$$

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[5] See (8.2) in "Construction. . . of the fractional calculus" (1997).

But

$$\begin{aligned}
D^\nu u &\equiv D\cdot D^{-(1-\nu)}u\\
&= D\cdot \frac{1}{\Gamma(2-\nu)}x^{1-\nu}\\
&= \frac{1}{\Gamma(1-\nu)}x^{-\nu}\\
D^{-(k-\nu)}u &= \frac{1}{\Gamma(1+k-\nu)}x^{k-\nu}
\end{aligned}$$

so we have

$$\begin{aligned}
D^\nu x^p &= \left\{\begin{bmatrix}\nu\\0\end{bmatrix}\frac{1}{\Gamma(1-\nu)} + \sum_{k=1}^{p}\begin{bmatrix}\nu\\k\end{bmatrix}\frac{p!}{(p-k)!}\frac{1}{\Gamma(1+k-\nu)}\right\}x^{p-\nu}\\
&= \sum_{k=0}^{p}\left\{\frac{\Gamma(1+\nu)}{\Gamma(1+k)\Gamma(1+\nu-k)}\frac{\Gamma(1+p)}{\Gamma(1+p-k)}\frac{1}{\Gamma(1+k-\nu)}\right\}x^{p-\nu}\\
&= \tfrac{\Gamma(1+p)}{\Gamma(1+p-\nu)}x^{p-\nu} \quad \text{according to } \textit{Mathematica} \qquad (12.2)
\end{aligned}$$

which we recognize to be precisely Lecroix' definition of the fractional derivative $D^\nu x^p$ $(0 < \nu < 1)$—recovered here as a corollary jointly of Lecroix' definition of the fractional integral $D^{-\nu}x^p$ and of the fractional generalization (10) of Leibniz' formula. If, on the other hand, we agree to accept Lecoix' definitions as given, then the preceding argument can be read (as I intended it) as a demonstration that (10) is consistent with the facts of the matter, and that the integrative aspects of Leibniz' formula—far from being a strange and unwelcome intrusion—are essential to its success.

The surprising fact is that while the statement

$$D^{\text{integer}}(f\cdot u) = D^{\text{integer}}(u\cdot f) \quad \text{is trivially \& uninformatively true}$$

its fractional analog/generalization

$$D^{\text{fraction}}(f\cdot u) = D^{\text{fraction}}(u\cdot f) \quad \text{is non-trivially \& informatively true} \qquad (13)$$

—at least so far as concerns functions of the type $f(x) = \sum f_p x^p$—and the same can be said in the more general case $f\cdot g$, as we have in fact had occasion already to see; the left and right sides of the latter of the preceding equalities lead to distinct computational problems, distinct organizations of the same functional material. And for none of this is there precedent in the ordinary calculus, which is (beyond the presence of all the Γ-functions) why exercise of the type that recently engaged us feels so unfamiliarly "strange."

3. Construction & applications of a "generalized fractional Leibniz formula". At (13) I identify one mechanism by which the fractional Leibniz formula invites one to play on-the-one-hand-this-but-on-the-other-hand-that. There exists also a second mechanism, which I undertake now to describe. By way of preparation...

Let A and B be real numbers. Trivially,

$$(A+B)^3, \quad A^3(1+A^{-1}B)^3, \quad (AB^{-1}+1)^3B^3$$

and more generally

$$A^p(A^{1-\frac{1}{3}p}B^{-\frac{1}{3}q}+A^{-\frac{1}{3}p}B^{1-\frac{1}{3}q})^3B^q$$

all yield $A^3+3A^2B+3AB^2+B^3$ when multiplied out. A similar remark pertains (for arbitrary p and q) to

$$(A+B)^N = A^p(A^{1-\frac{1}{N}p}B^{-\frac{1}{N}q}+A^{-\frac{1}{N}p}B^{1-\frac{1}{N}q})^NB^q$$

when $N = 0, 1, 2, \ldots$. Consider, however, the representative non-integral case $N = -1$: formally we have

$$(A+B)^{-1} = \begin{cases} A^{-1}\Big[1-\frac{B}{A}+\big(\frac{B}{A}\big)^2-\cdots\Big] \\ B^{-1}\Big[1-\frac{A}{B}+\big(\frac{A}{B}\big)^2-\cdots\Big] \end{cases}$$

which provide *distinct descriptions of the same expression*. But the first series is convergent if and only if $(B/A)^2 < 1$, and the second if and only if $(B/A)^2 > 1$; the series are in this sense complementary, and we do not imagine them to be *simultaneously* valid.

In (for example) quantum mechanics, where the general non-commutativity of operators A and B must be respected, we proceed rather differently to a somewhat different conclusion: from the triviality

$$\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}}(\mathsf{A}+\mathsf{B}-\mathsf{B})\frac{1}{\mathsf{A}+\mathsf{B}} = \frac{1}{\mathsf{A}} - \frac{1}{\mathsf{A}}\mathsf{B}\frac{1}{\mathsf{A}+\mathsf{B}}$$

we by iteration obtain[6]

$$(\mathsf{A}+\mathsf{B})^{-1} = \begin{cases} \mathsf{A}^{-1}\Big[\mathsf{I}-\mathsf{B}\mathsf{A}^{-1}+(\mathsf{B}\mathsf{A}^{-1})^2-\cdots\Big] \\ \mathsf{B}^{-1}\Big[\mathsf{I}-\mathsf{A}\mathsf{B}^{-1}+(\mathsf{A}\mathsf{B}^{-1})^2-\cdots\Big] \end{cases} \tag{14}$$

It is, in such a context, meaningless to impose convergence conditions of the form "$(\mathsf{B}/\mathsf{A})^2 < 1$." Such conditions attach instead to the number-valued inner products $(\varphi|\mathsf{B}/\mathsf{A}|\psi)$ that emerge in association with specific *applications* of (14); we are prepared to ascribe simultaneous formal validity to *both* variants of (14).

[6] See, for example, A. Massiah, *Quantum Mechanics* (1966), p. 715.

Returning in this light to (3), we find it natural to write

$$
\begin{aligned}
D^\nu(f\cdot g) = f\,\underbrace{(\overleftarrow{D}+\overrightarrow{D})^\nu}\,g \\
&= \overleftarrow{D}^\nu(1+\overleftarrow{D}^{-1}\,\overrightarrow{D})^\nu = \sum_{j=0}^{\infty}\begin{bmatrix}\nu\\ j\end{bmatrix}\overleftarrow{D}^{\nu-j}\,\overrightarrow{D}^j && (15.11)\\
&= \sum_{-\infty}^{\infty}\begin{bmatrix}\nu\\ j\end{bmatrix}\overleftarrow{D}^{\nu-j}\,\overrightarrow{D}^j && (15.12)\\
&= (\overleftarrow{D}\,\overrightarrow{D}^{-1}+1)^\nu\,\overrightarrow{D}^\nu = \sum_{k=0}^{\infty}\begin{bmatrix}\nu\\ k\end{bmatrix}\overleftarrow{D}^{k}\,\overrightarrow{D}^{\nu-k} && (15.21)\\
&= \sum_{-\infty}^{\infty}\begin{bmatrix}\nu\\ k\end{bmatrix}\overleftarrow{D}^{k}\,\overrightarrow{D}^{\nu-k} && (15.22)
\end{aligned}
$$

where we are dealing now with operators $\overrightarrow{D}$ and $\overleftarrow{D}$ that (by definition, as was noted already in connection with (3)) do commute, but which (because they are operators, not numbers) displace the locus of the convergence problem: convergence becomes contingent upon the structure of the operands, $f(x)$ and $g(x)$. Working from (15.11) with the aid of Lecroix' construction (12) we have

$$
\begin{aligned}
D^\nu(x^p\cdot x^q) &= \sum_{j=0}^{\infty}\begin{bmatrix}\nu\\ j\end{bmatrix}x^p\,\overleftarrow{D}^{\nu-j}\,\overrightarrow{D}^j x^q\\
&= \Big\{\sum_{j=0}^{\infty}\tfrac{\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+j)\Gamma(1+\nu-j)\Gamma(1+p-\nu+j)\Gamma(1+q-j)}\Big\}x^{p+q-\nu}
\end{aligned}
$$

Mathematica supplies

$$
\Big\{\text{etc.}\Big\} = \tfrac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}\Big[-\tfrac{\Gamma(-\nu)\Gamma(1+\nu)\sin\pi\nu}{\pi}\Big]\Big[-\tfrac{\Gamma(-q)\Gamma(1+q)\sin\pi q}{\pi}\Big]
$$

but appears to be unaware of the "reflection formula" $\frac{\Gamma(-x)\Gamma(1+x)\sin\pi x}{\pi}+1=0$ (see Spanier & Oldham's *Atlas of Functions* **43:5:1**), by virtue of which we achieve precise agreement with Lacroix' statement

$$
D^\nu x^{p+q} = \tfrac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}\,x^{p+q-\nu}
$$

Had we argued from (15.21) we would have achieved an identical result.

In an effort to proceed *directly* from (15.11) to (15.21) we set $k=\nu-j$ (therefore $j=\nu-k$) and use

$$
\begin{bmatrix}\nu\\ j\end{bmatrix} = \begin{bmatrix}\nu\\ \nu-k\end{bmatrix} = \begin{bmatrix}\nu\\ k\end{bmatrix}
$$

to obtain

$$\sum_{j=0}^{\infty}\begin{bmatrix}\nu\\j\end{bmatrix}\overleftarrow{D}^{\nu-j}\,\overrightarrow{D}^{j}=\sum_{\{k\}}\begin{bmatrix}\nu\\k\end{bmatrix}\overleftarrow{D}^{k}\,\overrightarrow{D}^{\nu-k}$$

where $\{k\}\equiv\{\nu,\nu-1,\nu-2,\ldots\}\hookrightarrow\{\ldots,\nu+2,\nu+1,\nu,\nu-1,\nu-2,\ldots\}$. In (15.21) the index k ranges, however, on $\{0,1,2,\ldots\}\hookrightarrow\{\ldots,-2,-1,0,+1,+2\}$, where $\hookrightarrow$ is intended to be read "which can, in the present context, be freely extended to the set." We are brought thus to the fairly remarkable conclusion that

$$x^p\cdot\sum_{\{k\}}\begin{bmatrix}\nu\\k\end{bmatrix}\overleftarrow{D}^{k}\,\overrightarrow{D}^{\nu-k}\cdot x^q=D^\nu x^{p+q}=\frac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}\,x^{p+q-\nu}$$

holds independently of whether we take k to range on

$$\{\ldots,-2\qquad,-1\qquad,0\qquad,+1\qquad,+2\qquad,\ldots\}$$

or on

$$\{\ldots,-2+\nu,-1+\nu,0+\nu,+1+\nu,+2+\nu,\ldots\}$$

This would become instantly intelligible if we could establish that k can in fact be taken to range on

$$\{\ldots,-2+\lambda,-1+\lambda,0+\lambda,+1+\lambda,+2+\lambda,\ldots\}$$

where the value of λ is *arbitrary*. We ask: Is it in fact the case that

$$x^p\cdot\sum_{\{j\}}\begin{bmatrix}\nu\\j+\lambda\end{bmatrix}\overleftarrow{D}^{j+\lambda}\,\overrightarrow{D}^{\nu-j-\lambda}\cdot x^q=\frac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}\,x^{p+q-\nu}\quad:\quad\text{all }\lambda \tag{16.1}$$

if j ranges on $\{\ldots,-2,-1,0,+1,+2,\ldots\}$? Can we, in other words, show that

$$\sum_{\{j\}}\Big\{\frac{1}{\Gamma(1+j+\lambda)\Gamma(1+\nu-j-\lambda)\Gamma(1+p-j-\lambda)\Gamma(1+q-\nu+j+\lambda)}\Big\} \tag{16.2}$$
$$=\frac{1}{\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)}\frac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}$$

For the convenience of *Mathematica* we write

$$\sum_{\{j\}}\Big\{\text{etc.}\Big\}=\sum_{0}^{\infty}\Big\{\text{etc.}\Big\}-\frac{1}{\Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)}+\sum_{-\infty}^{0}\Big\{\text{etc.}\Big\}$$

and are informed that

$$\sum_{0}^{\infty}\Big\{\text{etc.}\Big\}=\frac{\Gamma(-p+\lambda)\Gamma(\lambda-\nu)\sin[\pi(1+p-\lambda)]\sin[\pi(1-\lambda+\nu)]}{\pi^2\Gamma(1+\lambda)\Gamma(1+q+\lambda-\nu)}$$
$$\times\,{}_3F_2(1,-p+\lambda,\lambda-\nu;1+\lambda,1+q+\lambda-\nu;1)$$
$$\sum_{-\infty}^{0}\Big\{\text{etc.}\Big\}=\frac{\Gamma(-\lambda)\Gamma(-q-\lambda+\nu)\sin[\pi(1+\lambda)]\sin[\pi(1+q+\lambda-\nu)]}{\pi^2\Gamma(1+p-\lambda)\Gamma(1-\lambda+\nu)}$$
$$\times\,{}_3F_2(1,-\lambda,-q-\lambda+\nu;1+p-\lambda,1-\lambda+\nu;1)$$

where[7]

$$ {}_3F_2(A,B,C;a,b;x) \equiv \sum_{n=0}^{\infty} \frac{(A)_n(B)_n(C)_n}{(a)_n(b)_n} \frac{x^n}{n!} $$

is a "generalized hypergeometric function." The expressions $(A)_n$, $(B)_n$,... denote Pochhammer polynomials, of which I here repeat the definition:

$$ (P)_0 \equiv 1 \quad \text{and} \quad (P)_n \equiv P(P+1)(P+2)\cdots(P+n-1) $$

Drawing again upon the reflection formula

$$ \frac{\Gamma(-x)\sin[\pi(1+x)]}{\pi} = \frac{1}{\Gamma(1+x)} $$

we obtain

$$ \left. \begin{aligned} \sum_0^\infty \Big\{\text{etc.}\Big\} &= \tfrac{{}_3F_2(1,-p+\lambda,\lambda-\nu;1+\lambda,1+q+\lambda-\nu;1)}{\Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)} \equiv S^+(p,q,\nu;\lambda) \\ &\ \tfrac{1}{\Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)} \equiv S^{\,0}(p,q,\nu;\lambda) \\ \sum_{-\infty}^0 \Big\{\text{etc.}\Big\} &= \tfrac{{}_3F_2(1,-\lambda,-q-\lambda+\nu;1+p-\lambda,1-\lambda+\nu;1)}{\Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)} \equiv S^-(p,q,\nu;\lambda) \end{aligned} \right\} \tag{17} $$

Our problem, therefore, is to show that (for all λ)

$$ \begin{aligned} S^+(p,q,\nu;\lambda) - S^{\,0}(p,q,\nu;\lambda) + S^-(p,q,\nu;\lambda)& \\ = \tfrac{1}{\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)} \tfrac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}& \end{aligned} \tag{18} $$

As a check on the accuracy of my work, I set $\lambda = 0$ and obtain

$$ \begin{aligned} &S^+(p,q,\nu;0) = \text{desired expression} \\ &S^{\,0}(p,q,\nu;0) = S^-(p,q,\nu;0) = \tfrac{1}{\Gamma(1)\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q-\nu)} \end{aligned} $$

In this case, the leading term on the left side of (17) does all the work, while the remaining terms conspire to get out of the way. When, on the other hand, I set $\lambda = \nu$ the situation is reversed, since (transparently from (16.2)) only the leading term contributes to the S^+ sum; one obtains—whether one works directly from the sums or from the valuations assigned to them by (17)—

$$ \begin{aligned} &S^+(p,q,\nu;\nu) = S^{\,0}(p,q,\nu;\nu) = \tfrac{1}{\Gamma(1+\nu)\Gamma(1)\Gamma(1+p-\nu)\Gamma(1+q)} \\ &S^-(p,q,\nu;\nu) = \text{desired expression} \end{aligned} $$

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[7] See A. Erdélyi *et al*, *Higher Transcendental Functions* §5.1. Erdélyi's Chapters IV, V and VI provide a usefully detailed account of the theory of hypergeometric functions and their cognates, and have served as my principal source.

These results, while they provide a neat recapitulation of the experience that served initially to motivated me to ask questions (16), serve not at all to clarify what the answer to those questions (variants of the same question) might be. I pose those questions now in the form of a

$$\begin{aligned}\text{CONJECTURE:}\quad S(p,q,\nu;\lambda) &= S(p,q,\nu;0)\quad\text{for all }\lambda \\ &= \underbrace{\tfrac{1}{\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)}\tfrac{\Gamma(1+p+q)}{\Gamma(1+p+q-\nu)}}_{\text{confirmed easily by } \textit{Mathematica}}\end{aligned}\tag{19.1}$$

Here

$$\begin{aligned}S(p,q,\nu;\lambda) &\equiv S^{+}(p,q,\nu;\lambda) - S^{\,0}(p,q,\nu;\lambda) + S^{-}(p,q,\nu;\lambda)\\ &= \frac{N(p,q,\nu;\lambda)}{D(p,q,\nu;\lambda)}\end{aligned}$$

with

$$\begin{aligned}N(p,q,\nu;\lambda) &\equiv -1 + {}_3F_2(1,-p+\lambda,\lambda-\nu;1+\lambda,1+q+\lambda-\nu;1)\\ &\qquad + {}_3F_2(1,-\lambda,-q-\lambda+\nu;1+p-\lambda,1-\lambda+\nu;1)\\ D(p,q,\nu;\lambda) &\equiv \Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)\end{aligned}$$

The function $D(p,q,\nu;\lambda)$ has poles whenever one or another of the Γ-arguments hits one or another of the values $\{0,-1,-2,\ldots\}$. The validity of the conjecture requires that those be precisely counterbalanced by poles of $N(p,q,\nu;\lambda)$, and this appears to impose a burden upon the numerical resources of *Mathematica*. It becomes in this light attractive to work with the following reformulation of our conjecture

$$\frac{S(p,q,\nu;0)}{N(p,q,\nu;\lambda)} - \frac{1}{D(p,q,\nu;\lambda)} = 0 \tag{19.2}$$

which *Mathematica* does in fact appear to find much more digestible. Finally, we might write

$$\begin{gathered}N(p,q,\nu;\lambda) = S(p,q,\nu;0)D(p,q,\nu;\lambda)\\ \downarrow\end{gathered}$$

$$\begin{aligned}&{}_3F_2(1,-p+\lambda,\lambda-\nu;1+\lambda,1+q+\lambda-\nu;1) + {}_3F_2(1,-\lambda,-q-\lambda+\nu;1+p-\lambda,1-\lambda+\nu;1)\\ &\qquad = 1 + \frac{\Gamma(1+p+q)\Gamma(1+\lambda)\Gamma(1+\nu-\lambda)\Gamma(1+p-\lambda)\Gamma(1+q-\nu+\lambda)}{\Gamma(1+p+q-\nu)\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)}\end{aligned}\tag{19.3}$$

which is of analytical (rather than numeric) interest as a “gamma representation theorem” for the sum of hypergeometrics that stands on the left.

I am satisfied that the conjectured equations (19) are in fact true equations, but will postpone discussion of the evidence until after I have had opportunity to establish contact with the literature, and to explore certain. . .

4. Formal ramifications of the conjecture. Our conjecture is rooted in the formal expectation—to which it relates as a sufficient but not necessary condition—that $D^\nu(f\cdot g) = D^\nu(f\cdot g)$ should be not only true but *manifestly* true; it accomplishes that objective by permitting us (see again (16.1)) to write

$$D^\nu(f\cdot g) = f\cdot \sum_{\{j\}} \begin{bmatrix} \nu \\ j+\lambda \end{bmatrix} \overleftarrow{D}^{j+\lambda}\,\overrightarrow{D}^{\nu-j-\lambda}\cdot g \tag{20}$$

where $\{j\}\equiv\{\ldots,-2,-1,\quad 0,+1,+2,\ldots\}$ and λ is arbitrary.

Equation (20)—which gives back the classic result (10) at $\lambda=0$—is the subject of the first of the Osler papers (1970) cited in footnote 4, but was old already then; it had (as was brought to Osler's attention by a referee) been written down already by Y. Watanabe in 1931. Osler's basic motivation (and presumably also Watanabe's) was the same as my own, but his methods are usefully distinct from (and in many respects much more sophisticated than) mine.[8]

Osler has emphasized that (20) can be pressed into service as a kind of "identity generating machine," and has provided many examples of its use in that capacity; it will serve my illustrative purposes to describe just a few. By way of preparation, we remind ourselves that

$$\begin{aligned} D^\nu f(x) &= D\cdot D^{\nu-1}f(x) = D\cdot D^{-(1-\nu)}f(x)\\ &= D\cdot\frac{1}{\Gamma(1-\nu)}\int_0^x (x-y)^{(1-\nu)-1}f(y)\,dy\\ &= \frac{1}{\Gamma(-\nu)}\int_0^x (x-y)^{-\nu}f(y)\,dy \qquad (21)\\ &= \text{fractional integral of negative order (at least formally)} \end{aligned}$$

We will regard (20) as a statement about expressions of type (21), and will entrust to *Mathematica* the responsibility for doing the integrals.

$$\boxed{\text{Set } f(x) = \frac{x^{b-1}}{(1-x)^a},\ g(x) = \text{unit function},\ \nu=b-c,\ \lambda=0}$$

The left side of (20) then gives

$$\left[\frac{x^{c-1}\Gamma(b)}{\Gamma(c)}\right]{}_2F_1(a,b;c;x)$$

and it becomes clear that it was to achieve such a pretty result that Osler defined $f(x)$ as he did; his eye had evidently come to rest on entry **13.1.9** in

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[8] Osler is a mathematician, and pays a mathematician's careful attention to convergence criteria and other such niceties; as a physicist I am content—and in order to get from here to there in finite time generally prefer—to proceed much more formally, trusting to the physics itself to tell me when I have lapsed into material error.

the Erdélyi table of fractional integrals,[9] where he tinkered with the parameters so as to make the hypergeometric factor as simple as possible. Turning now to the right side of (20), we have

$$\begin{aligned}\sum_{\{j\}}\left\{\frac{x^{-1+b-j}\Gamma(b)}{\Gamma(b-j)}\,{}_2F_1(a,b;b-j;x)\right\}&\frac{\Gamma(1+b-c)}{\Gamma(1+j)\Gamma(1+b-c-j)}\\&\cdot\left\{\frac{x^{-b+c+j}}{(-b+c+j)\Gamma(-b+c+j)}\right\}\\=x^{c-1}\Gamma(b)\Gamma(1+b-c)\sum_{\{j\}}C(b,c;j)\,{}_2F_1(a,b;b-j;x)\end{aligned}$$

where

$$\begin{aligned}C(b,c;j)\equiv\underbrace{\frac{1}{\Gamma(1+b-c-j)\Gamma(-b+c+j)}}&\cdot\frac{1}{\Gamma(b-j)\Gamma(1+j)(-b+c+j)}\\=\frac{\sin\pi(-b+c+j)}{\pi}&=-(-)^j\frac{\sin\pi(b-c)}{\pi}\end{aligned}$$

Assembling and simplifying the results now in hand, we obtain

$${}_2F_1(a,b;c;x)=\tfrac{\Gamma(c)\Gamma(1+b-c)\sin\pi(b-c)}{\pi}\sum_{\{j\}}(-)^j\tfrac{1}{\Gamma(b-j)j!(b-c-j)}\,{}_2F_1(a,b;b-j;x)$$

We have here reproduced the first of the identities on Osler's list of twenty-one, by an argument which serves well enough to illustrate the "identity generation" potential of (20), and to make clear how Osler's notational conventions relate to my own. The example draws upon the fractional Leibniz formula in its classic form ($\lambda=0$). I turn now to an example which draws explicitly upon the feature of (20) which represents a generalization of that classic formula:

$$\boxed{\text{Set } f(x)=x^{c-b-1},\ g(x)=x^{d-a-1},\ \nu=c-a-1,\ \lambda=c-1}$$

The left side of (20) now gives

$$\frac{x^{-1-b+d}\Gamma(c+d-a-b-1)}{\Gamma(d-b)}$$

while the right side becomes

$$\begin{aligned}\sum_{\{j\}}\left\{\frac{x^{-b-j}\Gamma(c-b)}{\Gamma(1-b-j)}\right\}\frac{\Gamma(c-a)}{\Gamma(c+j)\Gamma(1-a-j)}\left\{\frac{x^{-1+d+j}\Gamma(d-a)}{\Gamma(d+j)}\right\}\\=x^{-1-b+d}\Gamma(c-b)\Gamma(c-a)\Gamma(d-a)\\\cdot\sum_{\{j\}}\frac{1}{\Gamma(1-a-j)\Gamma(1-b-j)\Gamma(c+j)\Gamma(d+j)}\end{aligned}$$

[9] *Tables of Integral Transforms*, Volume II, p.186.

But by the reflection formula

$$\frac{1}{\Gamma(1-a-j)\Gamma(1-b-j)}=\frac{\sin\pi a\sin\pi b\Gamma(a+j)\Gamma(b+j)}{\pi^2}$$

so from the results now in hand it follows that

$$\frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}=\frac{\sin\pi a\sin\pi b}{\pi^2}\sum_{\{j\}}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)\Gamma(d+j)}\tag{22}$$

We have here reproduced the tenth identity on Osler's list, which he calls "Dougall's formula," with reference to §1.4 of Erdélyi's *Higher Transcendental Functions*, Volume I. It is interesting to note that (20) gives rise to a line of argument which bears almost no resemblance to that outlined by Erdélyi.[10] And that *Mathematica*, when asked to sum the series, returns quite a different result:

$$\begin{aligned}\sum_{\{j\}}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)\Gamma(d+j)}&=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}\,{}_3F_2(1,a,b;c,d;1)\\&\quad+\frac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(c-1)\Gamma(d-1)}\,{}_3F_2(1,2-c,2-d;2-a,2-b;1)\end{aligned}$$

This, however, is a fairly immediate consequence of the *definition* of the hypergeometric function, and is therefore less richly informative than Dougall's formula.

5. Status of the conjecture. Looking again to the boxed assignments that led us to Dougall's formula (22), we set

$$\begin{aligned}c-b-1&=p\\d-a-1&=q\\c-a-1&=\nu\\c-1&=\lambda\end{aligned}$$

giving

$$\begin{aligned}a&=\lambda-\nu\\b&=\lambda-p\\c&=1+\lambda\\d&=1+q+\lambda-\nu\end{aligned}$$

[10] J. Dougall is cited several times by E. T. Whittaker in his *Modern Analysis*; he appears to have been a Scottish analyst, who published frequently on diverse topics in *Proc. Edinburgh Math. Soc.* during the first decades of the present century. His name, however, does not appear in any of the standard biographical sourcebooks.

in which notation (22) reads

$$\begin{aligned}
&\frac{\Gamma(1+p+q)}{\Gamma(1+\nu)\Gamma(1+p)\Gamma(1+q)\Gamma(1+p+q-\nu)}\\
&\qquad\qquad = \frac{\sin\pi(\lambda-\nu)\sin\pi(\lambda-p)}{\pi^2}\sum_{\{j\}}\frac{\Gamma(\lambda-\nu+j)\Gamma(\lambda-p+j)}{\Gamma(1+\lambda+j)\Gamma(1+q+\lambda-\nu+j)}\\
&\qquad\qquad = \sum_{\{j\}}\frac{1}{\Gamma(1-\lambda+\nu-j)\Gamma(1-\lambda+p-j)\Gamma(1+\lambda+j)\Gamma(1+q+\lambda-\nu+j)}
\end{aligned}$$

where to obtain the last equality we have appealed once again to the reflection formula and made use of $\sin\pi x = (-)^j \sin\pi(x+j)$. The result just achieved is precisely (16.2); our conjectured relation (19) is at this point recognized to be merely a *notational variant of Dougall's formula*. And Dougall's formula is—subject to the reported conditions

$$\Re(a+b-c-d) < 1 \quad : \quad a \text{ and } b \text{ not integers}$$

which in our notation read

$$\Re(p+q) > -1 \quad : \quad \lambda-\nu \text{ and } \lambda-p \text{ not integers}$$

—an established fact. So, therefore, is our conjecture. And so, finally, is (20), the superficially diverse consequences of which (see again Osler's list) are seen now all to be *disguised variants of Dougall's formula*.

Osler, in §4 of his paper, establishes (20) by an argument which makes heavy use of contour integral methods (and which, though Dougall's formula is not mentioned in this precise connection, must amount to a proof of that formula); his argument will win no beauty contests, but works.

I digress now to describe the upshot some recent work which, though rendered obsolete by the result just achieved, does seem to me to retain interest insofar as it suggests that "Dougall's formula is exquisitely delicate; though true, it is only *barely* true." I was led to that conclusion by the computational experience which I now summarize: To gain preliminary insight into the plausibility of (19.1) I proposed to assign "typical" values $\{\tilde{p},\tilde{q},\tilde{\nu}\}$ to the parameters $\{p,q,\nu\}$ and then—with the assistance of *Mathematica*—to plot $S(p,q,\nu;\lambda)$. Since $S(p,q,\nu;\lambda)$ is defined at (17) in terms of hypergeometric functions of unit argument, and since moreover we know that ${}_3F_2(A,B,C;a,b;z)$ is convergent everywhere (meaning "at all meaningful points") in parameter-space only if $|z|<1$, I elected (in order to be on the safe side) to make replacements of the form

$${}_3F_2(A,B,C;a,b;x) \quad \longleftarrow \quad {}_3F_2(A,B,C;a,b;1)$$

in (17) and to approach $x=1$ as a limit. The results achieved when I set $\{\tilde{p},\tilde{q},\tilde{\nu}\}=\{1,1,\frac{1}{2}\}$ appear to be typical; *Mathematica* reported

$$\begin{aligned}
S(1,1,\tfrac{1}{2};\lambda;x=\tfrac{999999}{1000000}) &= \frac{\text{sum of }{}_2F_1\text{ functions}\times\text{polynomials in }\lambda}{\text{product of }\Gamma\text{ functions}}\\
S(1,1,\tfrac{1}{2};\lambda;x=\tfrac{1000000}{1000000}) &= \frac{\text{sum of polygamma functions}\times\text{polynomials in }\lambda}{\text{product of }\Gamma\text{ functions}}\\
S(1,1,\tfrac{1}{2};\lambda;x=\tfrac{1000001}{1000000}) &= \frac{\text{former sum of }{}_2F_1\text{ functions}\times\text{polynomials in }\lambda}{\text{product of }\Gamma\text{ functions}}
\end{aligned}$$

and constructed the following figures:

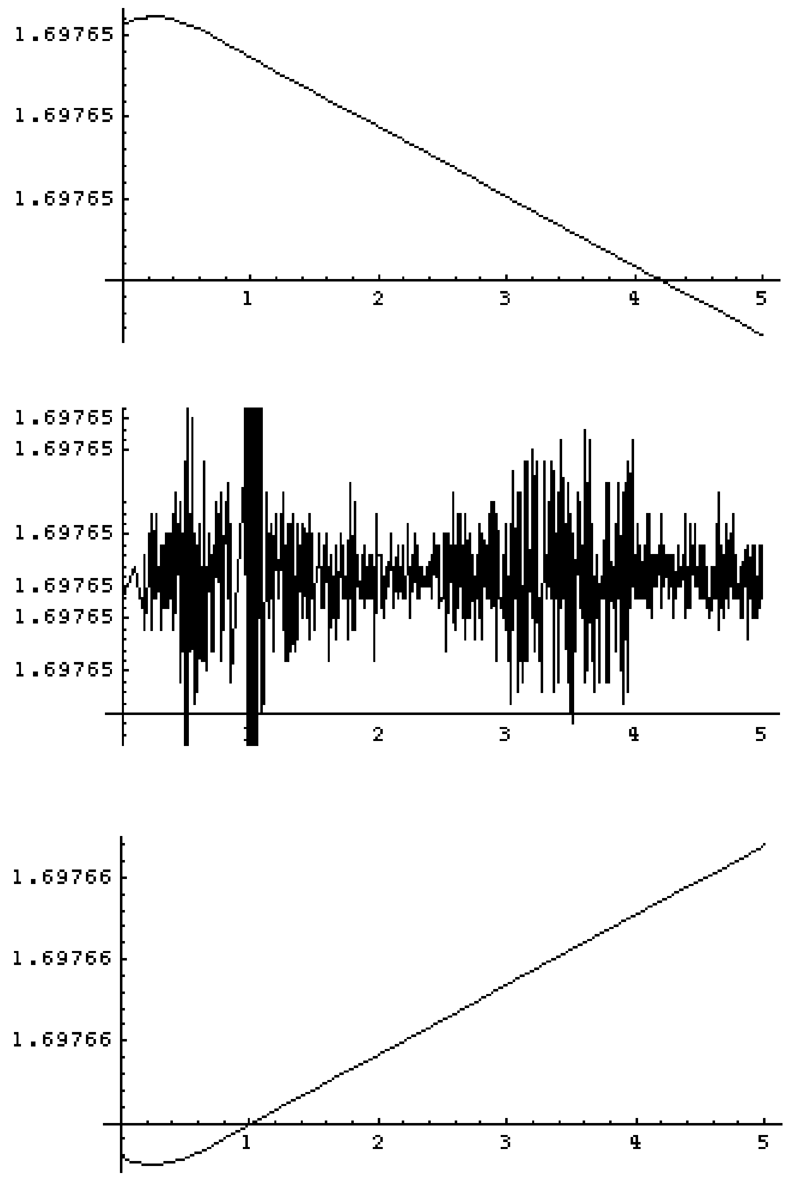

FIGURE 1: *Plots of the functions* $S(\tilde{p}, \tilde{q}, \tilde{\nu}; \lambda; x)$ *described in the text, with* $x = \frac{999999}{1000000}$ *(top),* $x = 1$ *(center),* $x = \frac{1000001}{1000000}$ *(bottom). The "chaos" evident at* $x = 1$ *appears to be scale-independent, in the sense that figures with* $0 \leq \lambda \leq \lambda_{\max}$ $(\lambda_{\max} = 0.001, 1, 100)$ *are qualitatively similar. The conjectured constant value is*

$$S(\tilde{p}, \tilde{q}, \tilde{\nu}; 0; 1) = \tfrac{16}{3\pi} = 1.69765$$

The upward/downward drift at $x = 1 \pm 10^{-6}$ (exaggerated in the figure) is slow enough to suggest that $S(\tilde{p}, \tilde{q}, \tilde{\nu}; \lambda; x \to 1)$ is destined to adhere constantly to the correct value, and provides no anticipatory hint of the radical qualitative adjustment seen at $x = 1$. The question therefore arose: Is the seeming chaos real (bad news for my conjecture!), or is it an artifact of the computational process? When I asked *Mathematica* to use 100-point precision in computing the values assumed by

$$f(\lambda) \equiv S(\tilde{p}, \tilde{q}, \tilde{\nu}; \lambda; 1) - S(\tilde{p}, \tilde{q}, \tilde{\nu}; 0; 1) \quad \text{at} \quad \lambda = \tfrac{n}{10} \quad : \quad n = 0, 1, 2, \ldots, 12$$

it complained repeatedly of "MaxExtraPrecision reached while evaluating..." but produced finally this data:

$$\begin{array}{l} \;\;0 \\ +0. \times 10^{-223} \\ +0. \times 10^{-223} \\ +0. \times 10^{-223} \\ +0. \times 10^{-223} \\ \;\;0 \\ -0. \times 10^{-223} \\ +0. \times 10^{-223} \\ -0. \times 10^{-223} \\ -0. \times 10^{-223} \\ \;\;0 \\ -0. \times 10^{-223} \\ -0. \times 10^{-223} \end{array}$$

While *Mathematica* found the variant (19.2) of our conjecture relatively more digestible, it did show clear symptoms of a similar gastric distress. I have, at present, nothing to say concerning either the origin or the ultimate significance of the instability brought thus to light.

6. Continuous analogs of the fractional Leibniz formula. Let (20) be notated

$$D^\nu(f \cdot g) = f \cdot \overleftrightarrow{D}(\nu; \lambda) \cdot g \tag{23}$$

Since the λ-parameterized operators $\overleftrightarrow{D}(\nu; \lambda)$ are (for ν given/fixed) functionally identical, they can be used in arbitrarily weighted linear combination:

$$D^\nu(f \cdot g) = f \cdot \sum_k w_k \overleftrightarrow{D}(\nu; \lambda_k) \cdot g \quad \text{provided} \quad \sum_k w_k = 1$$

which in the continuous limit reads

$$D^\nu(f \cdot g) = f \cdot \int w(\lambda) \overleftrightarrow{D}(\nu; \lambda)\, d\lambda \cdot g \quad \text{provided} \quad \int w(\lambda)\, d\lambda = 1$$

Reverting now to a more explicit notation, we have

$$\begin{aligned}\int w(\lambda)\overleftrightarrow{D}(\nu;\lambda)\,d\lambda &= \int w(\lambda)\sum_{\{j\}}\begin{bmatrix}\nu\\ \lambda+j\end{bmatrix}\overleftarrow{D}^{\lambda+j}\,\overrightarrow{D}^{\nu-(\lambda+j)}\,d\lambda\\ &= \iint w(\lambda)\sum_{\{j\}}\delta(\mu-j)\begin{bmatrix}\nu\\ \lambda+\mu\end{bmatrix}\overleftarrow{D}^{\lambda+\mu}\,\overrightarrow{D}^{\nu-(\lambda+\mu)}\,d\lambda d\mu\end{aligned}$$

The change of variables $\{\lambda,\mu\}\mapsto\{\lambda,\alpha\equiv\lambda+\mu\}$ has unit Jacobian, and permits us to write

$$\begin{aligned}&= \int\underbrace{\Big\{\int w(\lambda)\sum_{\{j\}}\delta(\alpha-\lambda-j)\,d\lambda\Big\}}_{\substack{=\sum_{\{n\}} w(\alpha+n)\\ \equiv\,\Omega(\alpha)}}\begin{bmatrix}\nu\\ \alpha\end{bmatrix}\overleftarrow{D}^{\alpha}\,\overrightarrow{D}^{\nu-\alpha}\,d\alpha\end{aligned}$$

where $\Omega(\alpha)$ has unit period $\Omega(\alpha+1)=\Omega(\alpha)$ and is otherwise subject only to the constraint

$$\int_0^1 \Omega(\alpha)\,d\alpha = 1$$

We are led thus to a *population* of "continuous analogs of the fractional Leibniz formula"

$$D^\nu(f\cdot g) = f\cdot\int_{-\infty}^{+\infty}\Omega(\alpha)\begin{bmatrix}\nu\\ \alpha\end{bmatrix}\overleftarrow{D}^{\alpha}\,\overrightarrow{D}^{\nu-\alpha}\,d\alpha\cdot g \tag{24}$$
$$\Omega(\alpha)\text{ any such periodic function}$$

Osler[11] has studied (24) in the special case $\Omega(\alpha)$ = unit function.

The periodic function $\Omega(\alpha)$ is the fruit of a train of thought which owes much to the spirit of the line of argument which led Poisson to the beautiful "Poisson summation formula."[12] It will be noted also that the intrusion of continuous methods into discrete problems is a phenomenon encountered a great variety of settings; for example, in the formal theory of (non-commutative) operator algebras one encounters[13] statements of the type

$$\tfrac{\partial}{\partial t}e^{\mathsf{A}(t)} = \int_0^1 e^{(1-u)\,\mathsf{A}(t)}\,\tfrac{\partial\mathsf{A}}{\partial t}\,e^{u\,\mathsf{A}(t)}\,du$$

[11] "The integral analog of the Leibniz rule," Math. Comp. **26**, 903 (1972).

[12] See pp. 75–77 of R. Courant & D. Hilbert, *Methods of Mathematical Physics* (1953) or P. Morse & H. Feshbach, *Methods of Theoretical Physics* (1953), pp. 466–467 & 483.

[13] See CLASSICAL MECHANICS (1983), pp. 287–287; R. Wilcox, "Exponential operators and parameter differentiation in quantum physics," J. Math. Physics **8**, 962 (1967).

and

$$[\mathsf{A}, e^{\mathsf{B}}] = \int_0^1 e^{(1-u)\,\mathsf{B}(t)}[\mathsf{A}, \mathsf{B}]e^{u\,\mathsf{B}(t)}\,du$$

where the effects of non-commutivity are, in effect, "smeared."

Equation (24) can be written

$$\frac{D^\nu(f\cdot g)}{\Gamma(1+\nu)} = \int_{-\infty}^{+\infty} \frac{D^\alpha f}{\Gamma(1+\alpha)}\cdot\frac{D^{\nu-\alpha}g}{\Gamma(1+\nu-\alpha)}\Omega(\alpha)\,d\alpha \tag{25}$$

where (see again (21) and make use once again of the reflection theorem)

$$\begin{aligned}\frac{D^\alpha f}{\Gamma(1+\alpha)} &= \int_0^x \frac{(x-y)^{-\alpha}f(y)}{\Gamma(1+\alpha)\Gamma(-\alpha)}\,dy\\ &= -\frac{\sin\pi\alpha}{\pi}\int_0^x (x-y)^{-\alpha}f(y)\,dy\end{aligned}$$

When viewed in this light (with eyes sufficiently de-focused), the architecture of (25) is seen to embody the general structure

$$\int FG = \int \{\textstyle\int F\}\{\textstyle\int G\}$$

and therefore to be reminiscent of a statement (Parseval's formula)

$$\int_{-\infty}^{+\infty} f(x)\tilde g(x)\,dx = \int_{-\infty}^{+\infty}\left\{\frac{1}{\sqrt{2\pi}}\int f(x)e^{ikx}\,dx\right\}\left\{\frac{1}{\sqrt{2\pi}}\int \tilde g(y)e^{-iky}\,dy\right\}dk$$

fundamental to the theory of the Fourier transform.[14] Osler has pursued in elaborate detail the fertile implications of this observation, and has produced many examples demonstrative of the fact that, just as (20) is a powerful "identity generating machine," so also—especially, but not exclusively, as they pertain to Fourier transform theory—are its continuous analogs (24). But I have no immediate need of such information, and my own (much more limited) objectives have been achieved. . . so I here abandon this pretty subject.

[14] See, for example, P. Morse & H. Feshbach, pp. 456–459.